

Continuity and Resurgence: towards a continuum definition of the \mathbb{CP}^{N-1} model

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We introduce a non-perturbative continuum framework to study the dynamics of quantum field theory (QFT), applied here to the \mathbb{CP}^{N-1} model, using Écalle's theory of resurgent trans-series, combined with the physical principle of continuity, in which spatial compactification and a Born-Oppenheimer approximation reduce QFT to quantum mechanics, while preventing all intervening rapid cross-overs or phase transitions. The reduced quantum mechanics contains the germ of all non-perturbative data, e.g., mass gap, of the QFT, all of which are calculable. For \mathbb{CP}^{N-1} , the results obtained at arbitrary N are consistent with lattice and large- N results. These theories are perturbatively non-Borel summable and possess the elusive IR-renormalon singularities. The trans-series expansion, in which perturbative and non-perturbative effects are intertwined, encapsulates the multi-length-scale nature of the theory, and eliminates all perturbative and non-perturbative ambiguities under consistent analytic continuation of the coupling. We demonstrate the cancellation of the leading non-perturbative ambiguity in perturbation theory against the ambiguity in neutral bion amplitudes. This provides a weak-coupling interpretation of the IR-renormalon, and a theorem by Pham *et al* implies that the mass gap is a resurgent function, for which resummation of the semi-classical expansion yields finite exact results.

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The 1970s-80s witnessed an intensive research program on two-dimensional (2D) asymptotically free non-linear sigma models, motivated by their relevance to antiferromagnetic spin systems and four dimensional (4D) QCD [1–6]. Many results were found in the large- N limit, particularly concerning dynamical mass generation and chiral symmetry breaking. However, concrete results in finite- N theories remain scarce [7], and even at large- N , the microscopic mechanism by which a mass gap is generated remains open to date. The inheritance from this époque is a list of deep problems/puzzles about \mathbb{CP}^{N-1} . A partial list includes: *i*) Invalidity of the dilute instanton gas approximation on \mathbb{R}^2 . In a theory in which the instanton has size moduli, the dilute instanton gas is ill-defined, since it assumes that the typical inter-instanton separation is much larger than instanton size. This is a variant of Coleman's "infrared embarrassment" problem. *ii*) Perturbation theory leads to a non-Borel-summable divergent series even after regularization and renormalization. This is a major reason why most mathematicians would still consider QFT non-rigorous. Attempts to Borel resum perturbation theory yield a class of ambiguities associated with singularities in the complex Borel plane. While some such ambiguities are cancelled by non-perturbative ambiguities (associated with 2D instanton-anti-instanton events), via a QFT version of the Bogomolny-Zinn-Justin mechanism [8, 9], there are other (more relevant) ambiguities associated with infrared (IR) renormalons [5, 6, 10, 11], and there are no known (semi-classical or otherwise) 2D configurations with which these ambiguities may cancel. Therefore, (Borel resummed) perturbation theory by it-

self is ill-defined. *iii*) Precise connection between large- N results and the instanton gas approximation [2–4, 12]. *iv*) Microscopic mechanism underlying the large- N mass gap for \mathbb{CP}^{N-1} : $m_g = \Lambda = \mu e^{-S_I/N} = \mu e^{-\frac{4\pi}{g^2 N}}$, where μ and Λ are the renormalization and strong scale.

In this Letter we revisit these problems using new physical and mathematical analytical tools. Our strategy is to use spatial (non-thermal) compactification and continuity to render the theory calculable, then apply Écalle's resurgence formalism [13] to unify perturbative and non-perturbative aspects in a manner that respects analytic continuation in the coupling, yielding exact unambiguous answers for physical observables.

The \mathbb{CP}^{N-1} model is described by a quantum field $n(x)$ in the coset space $\frac{U(N)}{U(N-1) \times U(1)}$. The action is

$$S = \int d^2x \left[\frac{2}{g^2} (D_\mu n)^\dagger D_\mu n - \frac{i\Theta}{2\pi} \epsilon_{\mu\nu} (D_\mu n)^\dagger D_\nu n \right] \quad (1)$$

where $D_\mu = \partial_\mu + iA_\mu$, A_μ is an auxiliary field, and Θ is the topological angle. \mathbb{CP}^{N-1} has a global $U(N)$ symmetry, and a local $U(1)$ gauge redundancy, $n \rightarrow e^{i\alpha(x)}n$. We parametrize \mathbb{CP}^{N-1} by locally splitting the n -field into a phase and modulus, $n_i = e^{i\varphi_i}r_i$, involving $(N-1)$ phase fields $\{\varphi_1, \dots, \varphi_N\}$, $\sum_{i=1}^N \varphi_i = 0 \pmod{2\pi}$ by $U(1)$ gauge redundancy, and $(N-1)$ modulus fields $\{r_1, \dots, r_N\}$, $\sum_{i=1}^N r_i^2 = 1$. We also consider adding N_f species of Dirac fermions, although here we mostly deal with the bosonic $N_f = 0$ theory.

Each $-\partial_\mu \varphi_i \equiv \mathcal{A}_{\mu,i}$ transforms as a "gauge" connection, $\mathcal{A}_{\mu,i} \rightarrow \mathcal{A}_{\mu,i} - \partial_\mu \alpha$, and we refer to it as the " σ -connection". With spatial compactification on $\mathbb{R} \times S_L^1$, we

define a new order parameter, the “ σ -connection holonomy”, making a circuit around the compact direction:

$$\begin{aligned} ({}^L\Omega)_j(x_1) &:= e^{i \int_0^L dx_2 \mathcal{A}_{2,j}} = e^{i(\varphi_j(x_1,0) - \varphi_j(x_1,L))} \\ {}^L\Omega(x_1) &:= \text{Diag} \left[({}^L\Omega)_1(x_1), \dots, ({}^L\Omega)_N(x_1) \right] \end{aligned} \quad (2)$$

Under an aperiodic global gauge rotation, the line operator ${}^L\Omega(x_1) \rightarrow e^{i \frac{2\pi k}{N}} {}^L\Omega(x_1)$, $k = 1, \dots, N$, reflecting the theory’s global \mathbb{Z}_N center-symmetry. (The fact that it is not $U(1)$ follows from the constraint, $\sum_{i=1}^N \varphi_i = 0 \pmod{2\pi}$, i.e., $\det {}^L\Omega(x_1) = 1$.) The matrix-valued gauge invariant σ -holonomy, ${}^L\Omega(x_1)$, plays an analogous role to the Wilson line in non-abelian $SU(N)$ gauge theory. Crucially, this operator contains more refined data than the familiar $U(1)$ Wilson line in the \mathbb{CP}^{N-1} model: $W = e^{i \int_0^L dx_2 A_2}$.

A typical classical background of the σ -connection holonomy is ${}^L\Omega_b = \text{Diag} [e^{2\pi i \mu_1}, e^{2\pi i \mu_2}, \dots, e^{2\pi i \mu_N}]$. Classically, this background is equivalent to imposing twisted boundary conditions for the \mathbb{CP}^{N-1} fields, of the form $n(x_1, x_2 + L) = {}^L\Omega_b n(x_1, x_2)$ [14, 15]. Undoing the twist by a field redefinition is equivalent to the substitution, $\partial_\mu \rightarrow \partial_\mu + i \delta_{\mu 2} \frac{2\pi}{L} \text{Diag} [\mu_1, \mu_2, \dots, \mu_N]$, analogous to turning on a Wilson line in compactified $SU(N)$ gauge theory (and further reason to call $-\partial_\mu \varphi_i = \mathcal{A}_{\mu,i}$ σ -connection). In the small- L weak coupling regime, the quantum mechanical stability of a given background can be determined via a one-loop analysis of the potential for the holonomy (2), similar to gauge theory. Integrating out weakly coupled Kaluza-Klein modes, we find

$$V_\pm[{}^L\Omega] = \frac{2}{\pi L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1 + N_f (\pm 1)^n) (|\text{tr } {}^L\Omega^n| - 1)$$

where $- (+)$ refers to thermal (spatial) compactification, where fermions have anti-periodic (periodic) boundary conditions. For $N_f \geq 0$, the N -fold degenerate minima in the thermal case are ${}^L\Omega_0^{\text{thermal}} = e^{i \frac{2\pi k}{N}} \mathbf{1}_N$, $k = 1, \dots, N$, a clumped configuration of holonomy eigenvalues. In sharp contradistinction, for $N_f \geq 1$ and in the spatial case, the minimum is unique, ${}^L\Omega_0^{\text{spatial}} = \text{Diag} \left(1, e^{i \frac{2\pi}{N}}, \dots, e^{i \frac{2\pi(N-1)}{N}} \right)$, a non-degenerate, \mathbb{Z}_N -symmetric holonomy, similar to QCD(adj) [16, 17]. The $N_f = 1$ case follows from non-perturbative effects [18].

Since there are no phase transitions (for finite- N) on $\mathbb{R} \times \mathbb{S}_L^1$, one may wonder what qualitative differences these two different backgrounds entail. In the thermal case, the potential at the minimum is the free energy density of the hot \mathbb{CP}^{N-1} model, $\mathcal{F} = V_-[{}^L\Omega_0^{\text{thermal}}] = -(2N - 2) \frac{\pi}{6} \left(1 + \frac{N_f}{2} \right) T^2 \sim O(N^1)$, the Stefan-Boltzmann result; whereas in the cold regime, $\mathcal{F} \sim O(N^0)$, because the spectral density of physical states is $O(N^0)$. There is a rapid cross-over from a hot deconfined regime to the cold

confined regime at the strong scale at finite- N , which becomes a sharp phase transition at $N = \infty$. With spatial compactification, the “free energy” density at small- L is $\mathcal{F} \sim O(N^0)$, just like the cold regime of the thermal theory. Therefore, there is no intervening rapid crossover (finite- N) or phase transition ($N = \infty$) as one dials the radius from large to small. This is the reason that the spatially compactified theory provides, in the small L regime, a weak-coupling semi-classical window into the confined regime [16, 17], whereas the thermally compactified theory provides, at small β , a weak coupling semi-classical window of the deconfined regime [4].

In the pure bosonic theory ($N_f = 0$) in which there is no distinction between the thermal and spatial compactification, a \mathbb{Z}_N -symmetric background is unstable. However, one can define a deformed bosonic \mathbb{CP}^{N-1} (analogous to deformed YM) by introducing heavy fermions $m \gg \Lambda$, so that the theory at distances larger than m^{-1} emulates the pure bosonic theory. Then, render the KK-modes sufficiently high such that the heavy fermions appear light with respect to the KK-modes, i.e., $m \lesssim \frac{2\pi}{LN}$. Thus, from the point of view of the one-loop potential, we can use the result of the massless theory, and at the same time, at distances larger than m^{-1} , the theory is the bosonic \mathbb{CP}^{N-1} model on a stable \mathbb{Z}_N -symmetric background.

In \mathbb{CP}^1 , in the small- L regime, the Lagrangian associated with the zero mode of the KK-tower is ($\xi \equiv \frac{2\pi}{LN}$):

$$S^{\text{zero}} = \frac{L}{2g^2} \int_{\mathbb{R}} (\partial_t \theta)^2 + \sin^2 \theta (\partial_t \phi)^2 + \xi^2 \sin^2 \theta, \quad (3)$$

where the kinetic term describes a particle on $S^2 \sim \mathbb{CP}^1$, and the potential follows from the \mathbb{Z}_2 stable background. This action has a semi-classical kink-instanton solution, which we call \mathcal{K}_1 , interpolating from $\theta = 0$ to $\theta = \pi$, and with action $S_1 = \frac{L}{g^2} (2\xi) = \frac{4\pi}{g^2} \times (\mu_2 - \mu_1) = \frac{S_I}{2}$. Its topological charge is $Q = \frac{1}{2}$. There is also an independent kink-instanton, \mathcal{K}_2 , with action $S_2 = \frac{S_I}{2}$, interpolating from $\theta = \pi$ to $\theta = 0$, also with topological charge $Q = \frac{1}{2}$. It is important to note that \mathcal{K}_2 is not the anti-kink $\bar{\mathcal{K}}_1$, which has $Q = -\frac{1}{2}$. The kink-instanton \mathcal{K}_2 is associated with the affine root of the $SU(2)$ algebra [14, 15].

This construction generalizes to \mathbb{CP}^{N-1} : there are N types of kinks, associated with the extended root system of the $SU(N)$ algebra, each of which carries a topological charge $Q = \frac{1}{N}$. The amplitude of \mathcal{K}_i has a non-perturbative factor

$$\mathcal{K}_i : e^{-S_0} = e^{-\frac{4\pi}{g^2 N}} = e^{-\frac{4\pi}{\lambda}} \sim e^{-\frac{S_I}{N}}, \quad i = 1, \dots, N \quad (4)$$

It is crucial to note the appearance of the ’t Hooft coupling, $g^2 N \equiv \lambda$, in the amplitude. Thus, the kink-instantons are exponentially more relevant than the 2D instanton: $e^{-\frac{4\pi}{g^2 N}} \gg e^{-\frac{4\pi}{g^2}}$.

At second order in the semiclassical expansion, there are self-dual and non-self-dual configurations. The non-

self-dual topological molecules are in one-to-one correspondence with the non-vanishing elements of the extended Cartan matrix \hat{A}_{ij} of $SU(N)$. For each entry $\hat{A}_{ij} < 0$ of the extended Cartan matrix, there exists a *charged bion*, $\mathcal{B}_{ij} \sim [\mathcal{K}_i \bar{\mathcal{K}}_j]$, which plays a crucial role in the mass gap of the theory with $N_f \geq 1$, similar to the magnetic bion in QCD(adj) on $\mathbb{R}^3 \times S^1$ [17]. For

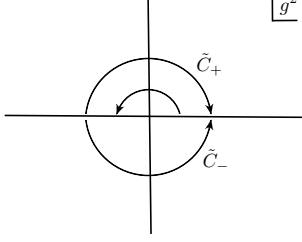


FIG. 1: Defining left (right) Borel sum $\mathbb{B}_{\theta=0^\pm}$, and left (right) neutral bion amplitude $[\mathcal{B}_{ii}]_{\theta=0^\pm}$. The $g^2 > 0$ line is a Stokes ray, the mathematical reason underlying the divergence of perturbation theory.

each diagonal entry, $\hat{A}_{ii} > 0$, there exists a *neutral bion*, $\mathcal{B}_{ii} \sim [\mathcal{K}_i \bar{\mathcal{K}}_i]$, with zero topological charge, and indistinguishable from the perturbative vacuum [19, 20]. The \mathcal{B}_{ii} generate a repulsion among the eigenvalues of the holonomy (2). For $g^2 > 0$, the constituents of the neutral bion interact attractively at short distances and the quasi-zero mode integral yields an amplitude which is naively meaningless. However, this is actually a reflection of the “Stokes phenomenon”. We can evaluate, in the $N_f = 0$ theory, the neutral bion amplitude as shown in Fig. 1. First, take $g^2 \rightarrow -g^2$, where the neutral bion amplitude is well-defined. Then, analytic continuation along \tilde{C}_\pm to $\theta = 0^\pm$ yields *left* and *right* amplitudes

$$[\mathcal{K}_i \bar{\mathcal{K}}_i]_{\theta=0^\pm} = \text{Re} [\mathcal{K}_i \bar{\mathcal{K}}_i] + i \text{Im} [\mathcal{K}_i \bar{\mathcal{K}}_i]_{\theta=0^\pm} \\ = \left(\log \left(\frac{\lambda}{8\pi} \right) - \gamma \right) \frac{16}{\lambda} e^{-\frac{8\pi}{\lambda}} \pm i \frac{16\pi}{\lambda} e^{-\frac{8\pi}{\lambda}} \quad (5)$$

The *absence* of a well-defined $\theta \rightarrow 0$ limit means that the semi-classical expansion by itself is also ill-defined. Écalle’s resurgent approach is to simultaneously apply this analytic continuation to the Borel summation of the perturbative sector *and* to the non-perturbative sector, in such a way that all ambiguities cancel, yielding an unambiguous and exact “trans-series” result. This mathematical technique has only been partially explored in QFT [20, 21], as most semi-classical studies only capture the first order. In quantum mechanics, this effect is studied in [8, 9]. Surprisingly, in QFT there appears to be universal behavior for the jump in the amplitude of neutral topological defects, arising from analytic continuation of the quasi-zero-mode integrals [18–20, 22].

The low energy Hamiltonian for (3), dropping ϕ -angle states in the Born-Oppenheimer approximation, is

$$H^{\text{zero}} = -\frac{1}{2} \frac{d^2}{d\theta^2} + \frac{\xi^2}{4g^2} [1 - \cos(2g\theta)] \quad (6)$$

The asymptotic form of the ground state energy, $\mathcal{E}_0(g^2)$, at large-orders in perturbation theory is evaluated in [23], using methods developed by Bender and Wu [24]:

$$\mathcal{E}_0(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_{0,q} (g^2)^q, \quad a_{0,q} \sim -\frac{2}{\pi} \left(\frac{1}{4\xi} \right)^q q! \quad (7)$$

The series is “Gevrey-1” [13], non-alternating, and hence non-Borel summable; a manifestation of the fact that we are expanding the ground state energy along a Stokes ray in the complex- g^2 plane. The Borel transform is given by $B\mathcal{E}(t) = -\frac{2}{\pi} \sum_{q=0}^{\infty} \left(\frac{t}{4\xi} \right)^q = -\frac{2}{\pi} \frac{1}{1-\frac{t}{4\xi}}$, and has a pole singularity on the positive real axis \mathbb{R}^+ (i.e., non-Borel summability or ambiguity of the sum). However, the series is *right*- and *left*-Borel resummable, given by $\mathcal{S}_{0^\pm} \mathcal{E}(g^2) = \frac{1}{g^2} \int_{C_\pm} dt B\mathcal{E}(t) e^{-t/g^2}$, where the contours C_\pm pass above (below) the singularity. Equivalently, taking $g^2 \rightarrow -g^2$, the series (7) becomes Borel summable. Analytically continuing the sum back along \tilde{C}_\pm yields the two “lateral” Borel sums:

$$\mathcal{S}_{0^\pm} \mathcal{E}(g^2) = \text{Re } \mathbb{B}_0 \mp i \frac{16\pi}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \quad (8)$$

where the imaginary part is the leading non-perturbative ambiguity of resummed perturbation theory, which is $O(e^{-S_{[\mathcal{K}_i \bar{\mathcal{K}}_i]}}) \sim e^{-2S_0} \sim e^{-2S_I/N}$, and $\text{Re } \mathbb{B}_0 \sim O(1)$ is the unambiguous real part.

We now come to the crux of the matter: the resummed vacuum energy has an imaginary part, but it is *not* associated with the Dyson *instability*, or *decay of the vacuum*. Rather, this ambiguous imaginary part is a direct reflection of the fact that (resummed) perturbation theory by itself is ill-defined. Furthermore, the ambiguity that we find for \mathbb{CP}^{N-1} on $\mathbb{R} \times S^1_L$ is parametrically the same as that of the elusive IR-renormalons. Therefore, what was viewed as a problem, in fact becomes a blessing in disguise: consider the Θ -independent part of the vacuum energy density in a trans-series expansion (combining perturbative and non-perturbative terms), and collect unambiguous terms and ambiguous terms together:

$$\mathcal{E}_{0,\text{transseries}}(g^2) \\ = \sum_{q=0}^{\infty} a_{0,q} (g^2)^q + [\mathcal{B}_{ii}] \sum_{q=0}^{\infty} a_{2,q} (g^2)^q + \dots (\text{formal}) \\ \longrightarrow \mathbb{B}_{0,\theta=0^\pm} + [\mathcal{B}_{ii}]_{\theta=0^\pm} \mathbb{B}_{2,\theta=0^\pm} + \dots (\text{BE} - \text{resummation}) \\ = \text{Re } \mathbb{B}_0 + \text{Re } [\mathcal{B}_{ii}] \text{Re } \mathbb{B}_2 + i [\text{Im } \mathbb{B}_{0,\theta=0^\pm} + \text{Re } \mathbb{B}_2 \text{Im } [\mathcal{B}_{ii}]_{\theta=0^\pm}] \\ = \text{Re } \mathbb{B}_0 + \text{Re } [\mathcal{B}_{ii}] \text{Re } \mathbb{B}_2, \quad \text{up to } e^{-4S_0} \quad (9)$$

In our explicit computation, we have taken $\mathbb{B}_2 = a_{2,0} + O(g^2)$, and kept only $a_{2,0}$ for consistency because we are also only accounting for the leading large orders asymptotics in (7). The sum of the left (right) Borel resummation of perturbation theory and non-perturbative left

(right) neutral bion amplitude is *unambiguous* at order $e^{-2S_0} = e^{-2S_I/N}$, as encoded in our perturbative-non-perturbative “confluence equation” in (9):

$$\text{Im } \mathbb{B}_{0,\theta=0^\pm} + \text{Re } \mathbb{B}_2 \text{Im } [\mathcal{B}_{ii}]_{\theta=0^\pm} = 0, \text{ up to } e^{-4S_0} \quad (10)$$

The passage from $\theta = 0^-$ to $\theta = 0^+$ is accompanied by a “Stokes jump” for the Borel resummation (8), which is mirrored by a jump in the neutral bion amplitude in the opposite direction (5) such that the sum of the two gives a unique result, with a smooth limit up to ambiguities at order e^{-4S_0} . Eq.(10) is conjectured to hold in (deformed) Yang-Mills in [19, 20], and here we verify it by explicit computation for \mathbb{CP}^{N-1} . Confluence equations are crucial for giving a non-perturbative continuum definition of QFT. We refer to the procedure in (9) as *Borel-Écalle resummation*, after Écalle’s seminal work [13], which formalized asymptotic expansions with exponentially small terms (trans-series) and generalized Borel resummation to account for the Stokes phenomenon.

As an application, we calculate a physically interesting non-perturbative quantity. The mass gap is the energy required to excite the system from the ground state to the first excited state. For \mathbb{CP}^1 , in the standard notation for Mathieu functions, the pair of states $ce_n(\theta, q)$ and $se_{n+1}(\theta, q)$ ($q = \frac{\xi^2}{4g^4}$), $n = 0, 1, 2, \dots$, become degenerate to all orders in perturbation theory: their asymptotic expansions are identical. As $g^2 \rightarrow 0$, the splitting $\mathcal{E}(b_{n+1}) - \mathcal{E}(a_n)$ is purely non-perturbative. The mass gap is defined as $m_g = \mathcal{E}(b_1) - \mathcal{E}(a_0)$ and is given by

$$m_g = \frac{8\pi}{g} \left(1 - \frac{7g^2}{16\pi} + O(g^4) \right) e^{-\frac{2\pi}{g^2}} \sim e^{-S_I/2} \quad (11)$$

This also justifies the Born-Oppenheimer approximation, because low-lying states are non-perturbatively split, whereas their separation from the higher states is an order one gap: $m_g \ll \mathcal{E}(a_1) - \mathcal{E}(b_1) \sim \Delta\mathcal{E}_\phi$, where $\Delta\mathcal{E}_\phi$ is the gap in the ϕ -sector in (3). For \mathbb{CP}^{N-1} , generalizing the above discussion, we find $m_g \sim \frac{1}{\sqrt{\lambda}} e^{-\frac{4\pi}{\lambda}} \sim e^{-S_I/N}$, which is a kink-instanton effect (4). We are not aware of any previous microscopic derivation in \mathbb{CP}^{N-1} of the all-important non-perturbative mass gap $\sim e^{-S_I/N}$. The gap at small- L may be considered as *the germ of the mass gap* for the theory on \mathbb{R}^2 . At large- N , this agrees with the mass gap obtained by the master field method [1].

One is entitled to ask whether this result is really meaningful, since there is a perturbative series multiplying the kink-instanton amplitude, see (11), which is itself a divergent asymptotic (non-Borel summable) series. This question has been answered in the mathematics literature. An important result by Pham et al, and Delabaere [25], using Écalle’s theory of resurgence [13], proves that the semi-classical expansions for the energy levels of the QM double-well and periodic potentials are indeed resurgent functions, resummable to finite, unambiguous, exact results. Our primary contribution here is

that we have found the conditions under which a non-trivial QFT such as \mathbb{CP}^{N-1} is connected to QM without any rapid cross-over or phase transition, i.e., by guaranteeing continuity. This permits us to derive the germ of all non-perturbative observables in QFT in the small- S^1 domain using rigorous QM results. Introducing the Θ dependence leads to a ‘grading’ of the resurgent trans-series structure [18]. The QFT results are in both qualitative and quantitative agreement with lattice and large- N results. We hope that this remarkable connection between QFT and QM may be used to explore other non-perturbative properties of general QFTs, and eventually lead to a fully consistent non-perturbative definition of non-trivial QFTs in the continuum.

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